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# UNPREDICTABILITY OF QUASI-PERIODIC DYNAMICAL SYSTEMS WITH FREQUENCY OF $p$ -ADIC LIOUVILLE TYPE NUMBERS

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## 1. INTRODUCTION

There are only two types of complete extensions of the rationals, the real numbers and the  $p$ -adic numbers. The  $p$ -adic numbers were first introduced by Kurt Hensel in 1897 and during almost 100 years they are considered mainly objects of pure mathematics. From 1980's applications of  $p$ -adic numbers were started and proposed in mathematical physics and quantum mechanics. Now  $p$ -adic analysis has been studied in various fields to investigate extremely complex models, which have chaotic properties.

In this paper we introduce the definitions of  $p$ -adic numbers and a special type of Schneider's continued fractions for  $p$ -adic numbers ([2], [5]). Then we study a quasi-periodic dynamical system with its frequency, which has a  $p$ -adic Liouville type property. We investigate the recurrent properties of its orbits by using these  $p$ -adic continued fractions and we estimate the positive gap values of the recurrent dimensions, which measure the unpredictability level of the orbit (cf. [8]).

Our plan of this paper is as follows. In section 2 we give a brief introduction of  $p$ -adic numbers. In section 3 we introduce  $p$ -adic continued fractions. In Section 4 we treat approximation lattices of  $p$ -adic numbers and we define  $p$ -adic weak Liouville numbers. In Section 5 we study the quasi-periodic dynamical system and in section 6 we give some numerical results on exponents of  $p$ -adic weak Liouville numbers by using an open-source mathematics software SAGE.

## 2. $p$ -ADIC NUMBERS

Here we give a brief introduction to  $p$ -adic numbers. (For further details, see [1], [3].)

For a prime  $p$  the  $p$ -adic valuation  $v_p$  is the function

$$v_p : \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{Z}$$

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defined as follows. For  $n \in \mathbb{Z}$ , let  $v_p(n)$  be the unique non-negative  $v$  such that

$$n = p^v \cdot n', \quad \gcd(p, n') = 1.$$

For  $x = a/b \in \mathbb{Q} \setminus \{0\}$ , define

$$v_p(x) = v_p(a) - v_p(b).$$

Then we can easily show the following properties

- (1)  $v_p(xy) = v_p(x) + v_p(y)$
- (2)  $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$

for all  $x, y \in \mathbb{Q} \setminus \{0\}$ .

For  $x \in \mathbb{Q}$  the  $p$ -adic absolute value  $|x|_p$  is defined as follows.

$$|x|_p = \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then the function  $x \rightarrow |x|_p$  satisfies the following conditions for all  $x, y \in \mathbb{Q}$ .

- (i)  $|x|_p = 0 \iff x = 0$
- (ii)  $|xy|_p = |x|_p |y|_p$
- (iii)  $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ .

Since  $\max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$ , the property (iii), called the strong triangle inequality, implies the triangle inequality

$$(iv) \quad |x + y|_p \leq |x|_p + |y|_p.$$

The absolute value, which satisfies the condition (i), (ii), (iv), is said to be Archimedean and if (iv) is replaced by the stronger (iii), then the absolute value is said to be non-Archimedean. The property (iii) implies the following isosceles triangle principle.

$$|x|_p \neq |y|_p \implies |x + y|_p = \max\{|x|_p, |y|_p\}$$

for all  $x, y \in \mathbb{Q}$ .

The property (iii) also implies the following equivalent relation for Cauchy sequences in  $\mathbb{Q}$ :

$$|x_n - x_m|_p \rightarrow 0 \text{ as } n, m \rightarrow \infty \iff |x_n - x_{n+1}|_p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The completion of  $\mathbb{Q}$  w.r.t.  $|\cdot|_p$  is called the field of  $p$ -adic numbers, denoted by  $\mathbb{Q}_p$ . The  $p$ -adic numbers  $x, y \in \mathbb{Q}_p$  satisfy the conditions (i), (ii), (iii). We define the ring of  $p$ -adic integers by

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

where  $x \in \mathbb{Z}_p$  has the expansion in base  $p$

$$x = \sum_{k=0}^{\infty} d_k p^k, \quad d_k \in \{0, 1, \dots, p-1\}.$$

### 3. $p$ -ADIC CONTINUED FRACTIONS

First we introduce a special type of Schneider's continued fractions, which was given by Bugeaud in [2] to investigate  $p$ -adic extremal numbers.

Let  $\{v_n\}$  be a sequence of positive integers. Then we consider the following  $p$ -adic continued fraction, which gives a  $p$ -adic number  $\omega$ :

$$\omega = 1 + \frac{p^{v_1}}{1 + \frac{p^{v_2}}{1 + \frac{p^{v_3}}{\ddots}}}$$

We denote the continued fraction by

$$\omega = 1 + [p^{v_1}/1, p^{v_2}/1, p^{v_3}/1, \dots].$$

On the other hand we can obtain the rational approximation  $\omega_n$  of  $\omega$  iteratively as follows.

$$\text{Let } P_{-1} = 1, Q_{-1} = 0, P_0 = 1, Q_0 = 1,$$

$$\begin{aligned} P_n &= P_{n-1} + p^{v_n} P_{n-2}, \\ Q_n &= Q_{n-1} + p^{v_n} Q_{n-2}, \quad n = 1, 2, \dots \end{aligned}$$

Then we call  $(P_n, Q_n)$  the  $p$ -adic convergents pair of  $p$ -adic number  $\omega$ . Then we have

$$\omega_n := P_n/Q_n = 1 + [p^{v_1}/1, p^{v_2}/1, \dots, p^{v_n}/1]$$

with the following estimates

$$|\omega - P_n/Q_n|_p = p^{-(v_1+v_2+\dots+v_{n+1})},$$

$$\max\{P_n, Q_n\} < p^{v_1+v_2+\dots+v_n}.$$

We need the following Lemma for the inequality relation between  $P_n$  and  $Q_n$ .

**Lemma 3.1.** *Given  $\omega \in \mathbb{Z}_p$ , let  $(P_n, Q_n)$  be the  $p$ -adic convergents pair of  $\omega$ . Then there exist positive constants  $c_1, c_2$ :*

$$c_1 Q_n \leq P_n \leq c_2 Q_n, \quad \forall n.$$

*Proof.* Using the  $p$ -adic continued fraction of  $\omega$ , we have

$$c_1 = p^{-v_2}, \quad c_2 = p^{v_1}.$$

In fact,

$$P_1 = p^{v_1}, \quad Q_1 = 1,$$

$$P_2 = p^{v_1}, \quad Q_2 = 1 + p^{v_2}$$

and the induction argument completes the proof.  $\square$

#### 4. APPROXIMATION LATTICES

In this section we introduce approximation lattices of  $p$ -adic numbers given by B. de Weger in [11]. Let  $p$  be a prime number and  $\alpha \in \mathbb{Z}_p$ . The ordered pair of rational integers  $(P, Q)$  is called a  $p$ -adic approximations to  $\alpha$  of order  $m$  if

$$|P - Q\alpha|_p = p^{-m}.$$

The set

$$\Gamma_m = \{(P, Q) \in \mathbb{Z}^2 : |P - Q\alpha|_p \leq p^{-m}\}$$

is called the  $m$ th approximation lattice of  $\alpha$ .

The approximation lattices satisfy the following properties.

- (i)  $\Gamma_m$  is a lattice in  $\mathbb{Z}^2$  of rank 2.
- (ii)  $\mathbb{Z}^2 = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_m \supset \Gamma_{m+1} \supset \cdots$
- (iii) A pair of points  $\{(P, Q), (R, S)\}$  in  $\Gamma_m$  is a basis of  $\Gamma_m$  if and only if  $|PS - QR| = p^m$ .

We consider the norm in  $\Gamma_m$ :

$$\|(P, Q)\| = \max\{|P|, |Q|\}.$$

First we choose the minimal non-zero point  $(X_1, Y_1)$  in  $\Gamma_m$  and secondly we choose the minimal point  $(X_2, Y_2)$  which is independent of  $(X_1, Y_1)$ . The pair of points  $\{(X_1, Y_1), (X_2, Y_2)\}$  is called the norm reduced basis.

We can show the following theorems by applying the results by Weger.

**Theorem 4.1.** *If  $(P, Q) \in \mathbb{Z}^2$  is the first minimal norm point, then it holds*

$$|P - Q\alpha|_p \leq \frac{1}{\|(P, Q)\|^2}.$$

**Theorem 4.2.** *If  $(P, Q) \in \mathbb{Z}^2$  is not the first minimal norm point in any lattice, then it holds*

$$|P - Q\alpha|_p > \frac{1}{2\|(P, Q)\|^2}.$$

Constructing an algorithm to search the minimal norm point in a general lattice is an extremely difficult problem, called SVP (shortest vector problem), which has direct applications to cryptography. Here we can use the algorithm for constructing  $p$ -adic continued fractions to find these minimal norm points.

To find these minimal points efficiently we define the following restrictions of the lattices. We denote

$$\begin{aligned} \Gamma_m^{/c_1} &= \{(P, Q) \in \Gamma_m : c_1 Q \leq P\}, \\ \Gamma_m^{c_2/} &= \{(P, Q) \in \Gamma_m : P \leq c_2 Q\}, \\ \Gamma_m^{c_2/c_1} &= \{(P, Q) \in \Gamma_m : c_1 Q \leq P \leq c_2 Q\} \end{aligned}$$

for some constants  $c_2 > c_1 > 0$ .

In section 3 we can iteratively construct the convergents  $(P_n, Q_n)$ , using  $p$ -adic continued fractions of  $\omega$ :

$$|P_n - Q_n \omega|_p = p^{-(v_1 + \cdots + v_{n+1})},$$

$$\|(P_n, Q_n)\| < p^{v_1 + \dots + v_n}.$$

For a sequence of the first minimal norm points we can choose the subsequence of  $\{(P_{m_j}, Q_{m_j})\}$  from the  $p$ -adic convergents pairs under some algebraic assumptions on  $p$ -adic numbers. In our previous papers ([7],[9]) we introduced a class of irrational numbers, which have well rational approximation properties.

In  $p$ -adic numbers case we also say that a  $p$ -adic number  $\omega$  is a weak Liouville number with its order  $\gamma > 0$  if there exists a subsequence  $\{(P_{m_j}, Q_{m_j})\}$  which satisfies

$$(4.1) \quad |P_{m_j} - Q_{m_j}\omega|_p \leq \frac{1}{\|(P_{m_j}, Q_{m_j})\|^{2+\gamma}}.$$

Assume that  $v_{n+1} > v_1 + \dots + v_n + 2c$  for a constant  $c : 2p^{-2c} < 1$ . Then we have

$$(4.2) \quad \begin{aligned} |P_n - Q_n\omega|_p &\leq p^{-2(v_1 + \dots + v_n + c)} \\ &\leq \frac{1}{2\|(P_n, Q_n)\|^2}. \end{aligned}$$

It follows from Theorem 4.2 that  $(P_n, Q_n)$  is the first minimal norm point in  $\Gamma_n$ .

If we assume that  $v_{n+1} > (1 + \gamma)(v_1 + \dots + v_n)$  for  $\gamma > 0$ , then we have

$$\begin{aligned} |P_n - Q_n\omega|_p &\leq p^{-(2+\gamma)(v_1 + \dots + v_n)} \\ &\leq \frac{1}{\|(P_n, Q_n)\|^{2+\gamma}}. \end{aligned}$$

## 5. QUASI-PERIODIC ORBITS

For a given  $\omega \in \mathbb{Z}_p$  and  $Q \in \mathbb{N}_0$  we denote

$$\|Q\omega\|_p = \min\{|P - Q\omega|_p : c_1Q \leq P \leq c_2Q, P \in \mathbb{N}_0\}$$

where  $c_1, c_2$  are the constants in Lemma 3.1. We denote a minimal pair by  $(P_Q, Q)$ :

$$|P_Q - Q|_p = \min\{|P - Q\omega|_p : c_1Q \leq P \leq c_2Q, P \in \mathbb{N}_0\}.$$

We consider the shift mapping  $f$  on  $\mathbb{Z}_p$  by

$$f(x) = x + \omega, \quad x \in \mathbb{Z}_p.$$

We investigate some recurrent property of the orbit  $\Sigma_x$  given by  $f$ ,

$$\Sigma_x = \{x, f(x), f^2(x), \dots, f^n(x), \dots\}$$

where  $f^2(x) = f(f(x))$ ,  $f^3(x) = f(f(f(x)))$ ,  $\dots$ . We have

$$f^n(x) = x + n\omega$$

and hence

$$\|f^n(x) - x\|_p = \|n\omega\|_p.$$

We estimate the recurrent properties of the q.p. orbits and the gap of recurrent dimensions by using the algebraic property,  $p$ -adic weak Liouville property, of the irrational frequency  $\omega$ .

**Definitions of recurrent dimensions:**

Define the first  $\varepsilon$ -recurrent time by

$$M_\omega(\varepsilon) = \min\{m \in \mathbb{N} : \|f^m(x) - x\|_p = \|m\omega\|_p \leq \varepsilon\}.$$

The upper recurrent dimension is defined by

$$\overline{D}_\omega = \limsup_{\varepsilon \rightarrow 0} \frac{\log M_\omega(\varepsilon)}{-\log \varepsilon}$$

and the lower recurrent dimension is defined by

$$\underline{D}_\omega = \liminf_{\varepsilon \rightarrow 0} \frac{\log M_\omega(\varepsilon)}{-\log \varepsilon}.$$

Then we can define the gaps of recurrent dimensions by  $G_\omega = \overline{D}_\omega - \underline{D}_\omega$ .

If the gap values  $G_\omega$  take positive values, we cannot exactly determine or predict the  $\varepsilon$ -recurrent time of the orbits:

$$M_\omega(\varepsilon) \in \left[ \left( \frac{1}{\varepsilon} \right)^{\underline{D}_\omega}, \left( \frac{1}{\varepsilon} \right)^{\overline{D}_\omega} \right].$$

Thus we propose the value  $G_\omega$  as the parameter, which measures the unpredictability level of the orbit (cf. [8]).

We obtain the following theorem.

**Theorem 5.1.** *Let  $\omega$  be a weak Liouville number with its order  $\gamma > 0$ , which satisfies (4.1). Then we have*

$$\underline{D}_\omega \leq \frac{1}{2 + \gamma}.$$

*Proof.* For the sequence  $\{(P_{m_j}, Q_{m_j})\}$ , which specifies the weak Liouville property of  $\omega$ ;

$$|P_{m_j} - Q_{m_j}\omega|_p \leq \frac{1}{\|(P_{m_j}, Q_{m_j})\|^{2+\gamma}},$$

it follows from Lemma 3.1 that we have

$$c_1 Q_{m_j} \leq \|(P_{m_j}, Q_{m_j})\| \leq c_2 Q_{m_j}.$$

It follows from the definitions that

$$\begin{aligned} \|f^{Q_{m_j}}x - x\|_p &= \|Q_{m_j}\omega\|_p \\ &\leq |P_{m_j} - Q_{m_j}\omega|_p \\ &\leq \frac{1}{\|(P_{m_j}, Q_{m_j})\|^{2+\gamma}}. \end{aligned}$$

Thus we have

$$\|f^{Q_{m_j}}x - x\|_p \leq \frac{1}{c_1^{2+\gamma} Q_{m_j}^{2+\gamma}} := \varepsilon_j.$$

Now we can estimate the lower recurrent dimension as follows.

$$\begin{aligned}
\underline{D}_\omega &= \liminf_{\varepsilon \rightarrow 0} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\
&= \liminf_{j \rightarrow \infty} \inf_{\varepsilon_{j+1} \leq \varepsilon \leq \varepsilon_j} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\
&\leq \liminf_{j \rightarrow \infty} \frac{\log M(\varepsilon_j)}{-\log \varepsilon_j} \\
&\leq \lim_{j \rightarrow \infty} \frac{\log Q_{m_j}}{-\log c_1 + (2 + \gamma) \log Q_{m_j}} \\
&= \frac{1}{2 + \gamma}.
\end{aligned}$$

□

For the upper estimate we assume some conditions on the sparse between  $(P_{m_j}, Q_{m_j})$  and  $(P_{m_{j-1}}, Q_{m_{j-1}})$ .

**Theorem 5.2.** *Under the same hypotheses as Theorem 5.1 we assume that there exist a sequence of lattice points  $(P_{l_j}, Q_{l_j}) \in \Gamma_{l_j}^{c_2/c_1}$ :  $Q_{m_{j-1}} < Q_{l_j} < Q_{m_j}$ ,  $\forall j$ , which are not the first minimal points in any lattice, and a sequence of constants  $K_j$ :  $0 < K_j < 1$ , which satisfy*

$$|P - Q\omega|_p \geq \frac{K_j}{\|(P_{l_j}, Q_{l_j})\|^2}$$

for every  $(P, Q) \in \Gamma_m^{c_2/c_1}$ :  $Q_{m_{j-1}} \leq Q < Q_{l_j}$ . Then we have

$$\overline{D}_\omega \geq \frac{1}{2}.$$

Consequently, we can estimate the positive gap value

$$G_\omega \geq \frac{\gamma}{2(2 + \gamma)}.$$

*Proof.* It follows from Theorem 4.2 that we have

$$\begin{aligned}
|P_{l_j} - Q_{l_j}\omega|_p &\geq \frac{K_j}{2\|(P_{l_j}, Q_{l_j})\|^2} \\
&\geq \frac{1}{cQ_{l_j}^2} := \varepsilon_j.
\end{aligned}$$

For  $Q \in \mathbb{N}$ :  $Q_{m_{j-1}} \leq Q < Q_{l_j}$ , take the minimal pair  $(P_Q, Q)$ :

$$|P_Q - Q\omega|_p = \min\{|P - Q\omega|_p : c_1Q \leq P \leq c_2Q, P \in \mathbb{N}_0\}.$$

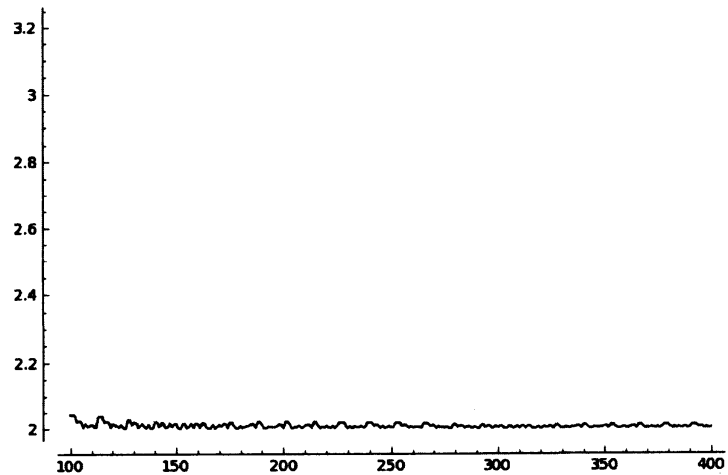
Then we have

$$\begin{aligned}
|P_Q - Q\omega|_p &\geq \frac{K_j}{\|(P_{l_j}, Q_{l_j})\|^2} \\
&\geq \frac{1}{cQ_{l_j}^2} = \varepsilon_j.
\end{aligned}$$





Quadratic irrational case:  $\sqrt{29} \in \mathbb{Z}_p$ ,  $p = 5$ ,  $m = 100 \sim 400$  (cf.[6])



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